DYNAMICAL LOCALIZATION OF DIRAC PARTICLES IN ELECTROMAGNETIC FIELDS WITH DOMINATING MAGNETIC POTENTIALS

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ABSTRACT. We consider two-dimensional massless Dirac operators in a radially symmetric electromagnetic field. In this case the fields may be described by one-dimensional electric and magnetic potentials V and A. We show dynamical localization in the regime when $\lim_{r\to\infty} |V|/|A| < 1$, where dense point spectrum occurs.

1. INTRODUCTION

Graphene is a two dimensional material consisting of carbon atoms arranged in a honeycomb lattice which was isolated in 2004 [14]. Behind its remarkable properties such as Klein tunneling and finite minimal conductivity [9] stays the fact that at low excitations energies the dynamics of charge carriers is described by the massless two-dimensional Dirac operator [3]. For technological devices based on graphene one needs the ability to confine and control the mobility of charge carriers. However, confining Dirac particles is not an easy task due the so-called Klein effect, where particles are able to penetrate electric potential walls [9] with very little reflexion index. In [7] it was argued that in presence of rotational symmetric electric and magnetic fields one could confine or deconfine Dirac particles by manipulating the strength of the fields at infinity, i.e., far away from the sample. Our main result is a dynamical statement on this effect and a continuation of a recent work [11] by two of the present authors. Before presenting the result let us explain this phenomenon with more mathematical details.

Denote by H the two-dimensional massless Dirac operator coupled to a radially symmetric field $\mathbf{E} = E\hat{r}$ on the plane and a radially symmetric transversal magnetic field B. If the fields are sufficiently regular H is a self-adjoint operator densely defined in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ acting as

$$H = \boldsymbol{\sigma} \cdot (-i\nabla - \mathbf{A}) + V, \tag{1}$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is the electric potential satisfying

$$V(\mathbf{x}) = -\int_0^{|\mathbf{x}|} E(s)ds \equiv V(r)$$

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(abusing notation, we write $V(r)$ to denote $V(\mathbf{x})$, where $r = |\mathbf{x}| \in [0, \infty)$ is the standard radial variable). Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ is a matrix-valued vector whose components are the first two Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The vector potential $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ generates the magnetic field B, with $B = \partial_1 A_2 - \partial_2 A_1$. We choose the rotational gauge, i.e., we set

$$\mathbf{A}(\mathbf{x}) := \frac{1}{r} A(r) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \text{with} \quad A(r) = \frac{1}{r} \int_0^r B(s) s \mathrm{d}s.$$

We note that, besides some local regularity requirements for (V, A), H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ independently of the growth rate of A and V (see [4]).

In this setting there exists a unitary transform [18] (see also [10, Section 6])

$$\mathcal{U} : L^2(\mathbb{R}^2, \mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^+, \mathbb{C}^2) \otimes \ell^2(\mathbb{Z}), \tag{2}$$

such that the operator H can be written as a direct sum of operators on the half-line

$$\mathcal{U} H \mathcal{U}^* = \bigoplus_{j \in \mathbb{Z}} h_j, \tag{3}$$

where

$$h_j = -i\sigma_2 \partial_r + \sigma_1 (A - \tfrac{m_j}{r}) + V \quad \text{on} \quad L^2(\mathbb{R}^+, \mathbb{C}^2), \tag{4}$$

with $m_j = j + \frac{1}{2}$ for $j \in \mathbb{Z}$. Clearly, the spectra of H and h_j are related through

$$\sigma(H) = \overline{\bigcup_{j \in \mathbb{Z}} \sigma(h_j)}, \quad \sigma_c(H) = \overline{\bigcup_{j \in \mathbb{Z}} \sigma_c(h_j)}, \quad \text{and} \quad \sigma_{pp}(H) = \bigcup_{j \in \mathbb{Z}} \sigma_{pp}(h_j). \tag{5}$$

For the operators h_j we have the following properties, assuming sufficiently regular fields: If $A(r) \to \infty$ as $r \to \infty$ and

$$\overline{\lim_{r \to \infty}} \left| \frac{V(r)}{A(r)} \right| < 1, \tag{6}$$

then the spectrum of h_j is discrete for each $j \in \mathbb{Z}$ (see [12, Proposition 1] for the precise regularity conditions). In contrast, if $V(r) \to \infty$ as $r \to \infty$ and

$$\overline{\lim_{r \to \infty}} \left| \frac{A(r)}{V(r)} \right| < 1, \tag{7}$$

then the spectrum of h_j equals the whole real line and it is purely absolutely continuous [16, Propsition 2]. This suggests delocalized particles in the regime given by (7) and confined particles in the one given by (6). However, the latter is not obvious since, for fields satisfying (6), H may have dense point spectrum (see [13] and [18, Theorem 7.10] for the case when B decays at infinity and [12] for the case when B is not decaying at infinity). We recall that dense point spectrum may lead to non-trivial dynamics. In fact, in this case, it is only known that the wave-packet spreading is sub-ballistic [17] (the result stated in [17] is for Laplace-type operators but can easily be adapted for the Dirac case). Moreover, there are examples of systems with pure point spectrum where the spreading rate is arbitrarily close to the ballistic one [5, 6] (see also [1]).

Concerning dynamical results we know that particles in electromagnetic fields satisfying (7) behave ballistically, i.e., for any finite energy state $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ and $\kappa > 0$ one has (see [11])

$$\frac{1}{T}\int_0^\infty \left\| |\mathbf{x}|^{\kappa/2} \mathrm{e}^{-iHt}\psi \right\|^2 dt \sim T^\kappa, \quad \text{for large} \quad T > 0.$$

The main result of this work is that under condition (6) the operator H exhibits dynamical localization, i.e, for any $\kappa > 0$, for any finite energy interval I, and for any state $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$, with sufficient regularity in the angular variable (depending on κ; see (12)), holds

$$\sup_{t\geq 0} \left\| |\mathbf{x}|^{\kappa/2} \ \mathrm{e}^{-itH} P_I(H)\psi \right\|^2 < \infty,$$

where $P_I(H)$ is the spectral projection of H onto I.

Let us now state our assumptions and result more precisely:

Hypothesis 1. *$A, V \in C^1(\mathbb{R}^+, \mathbb{R})$ and they satisfy*

(8) $$|A(r)| \to \infty \quad \text{as} \quad r \to \infty,$$

(9) $$\overline{\lim_{r\to\infty}} \left| \frac{V(r)}{A(r)} \right| < 1,$$

(10) $$\lim_{r\to\infty} \left| \frac{A'(r)}{A^2(r)} \right| = 0.$$

Recall that $\mathcal{U}$ (see (2)) is the unitary map that decomposes H in the direct sum of the operators h_j. For a given $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ we write

(11) $$\mathcal{U}\psi = \bigoplus_{j\in\mathbb{Z}} \varphi_j, \quad \text{with} \quad \varphi_j \in L^2(\mathbb{R}^+, \mathbb{C}^2).$$

Our main result is the following theorem.

Theorem 1.1. *Let $\kappa > 0$, $I \subset \mathbb{R}$ be a bounded energy interval and let $P_I(H)$ be the spectral projection of H onto I. Assume that A and V fulfill Hypothesis 1 and let $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ be a normalized state such that $\mathcal{U}\psi = \bigoplus_{j\in\mathbb{Z}} \varphi_j$ satisfies*

(12) $$\sum_{j\in\mathbb{Z}} |j|^\kappa \, \|\varphi_j\|^2 < \infty.$$

Then we have

(13) $$\sup_{t\geq 0} \left\| |\mathbf{x}|^{\kappa/2} \ \mathrm{e}^{-itH} P_I(H)\psi \right\|^2 < \infty.$$

We note that the condition (12) is related to regularity of the initial state ψ in the angular variable. Indeed, let $r^{-1/2}\tilde{\psi}$ with $\tilde{\psi} \in L^2(\mathbb{R}^+ \times [0, 2\pi), \mathbb{C}^2)$ be equal to $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ expressed in polar coordinates. Then, (12) follows if $\tilde{\psi} \in H^{\kappa/2}([0, 2\pi), L^2(\mathbb{R}^+)) \oplus H^{\kappa/2}([0, 2\pi), L^2(\mathbb{R}^+))$. Here, for $\kappa > 0$

$$H^{\kappa/2}\left([0, 2\pi), L^2(\mathbb{R}^+)\right) := \{u \in L^2([0, 2\pi) \times \mathbb{R}^+), \sum_{\ell\in\mathbb{Z}} (1 + |\ell|)^\kappa \, \|\hat{u}_\ell\|^2_{L^2(\mathbb{R}^+)} < \infty\},$$

is the fractional Sobolev space on the torus [8] and $\hat{u}_\ell$ is the $\ell-$th Fourier coefficient of u with respect to the variable θ. To be more explicit, note that $\mathcal{F}\tilde{\psi} = \bigoplus_j \varphi_j$ where,

for $g \in L^2([0,2\pi),\mathbb{C}^2)$,

$$(\mathcal{F}g)(j) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} M_\theta \ \mathrm{e}^{-im_j\theta} g(\theta)\mathrm{d}\theta, \quad \text{and} \quad M_\theta = \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & i\mathrm{e}^{-i\theta/2} \end{pmatrix}.$$

We notice that

$$(\mathcal{F}g)(j) = \begin{pmatrix} \hat{g}_1(j) \\ i\hat{g}_2(j+1) \end{pmatrix}.$$

The angular momentum operator $J := -i\partial_\theta + \sigma_3/2$ satisfies $\mathcal{F}J\mathcal{F}^* = m_j$. Assuming for simplicity that $\kappa = 2n$ with $n \in \mathbb{N}$, we have

$$\Big(\sum_{j\in\mathbb{Z}} |m_j|^{2n} \,\|\varphi_j\|^2\Big)^{1/2} = \left\|J^n \tilde{\psi}\right\| = \left\|(-i\partial_\theta + \frac{\sigma_3}{2})^n \tilde{\psi}\right\|$$
$$\leq \sum_{k=0}^{n} \binom{n}{k} \left\|(-i\partial_\theta)^k \tilde{\psi}\right\| \leq 2^n \left\|\tilde{\psi}\right\|_{H^n([0,2\pi),L^2(\mathbb{R}^+))^2}.$$

We can also derive a sufficient condition on ψ for (12) to hold. To avoid unnecessary complications, we stick to the case of even values of κ. Since $\partial_\theta = -x_2\partial x_1 + x_1\partial x_2$, we have

$$\left\|(\partial_\theta)^k \tilde{\psi}\right\|^2 = \left\|(-x_2\partial_{x_1} + x_1\partial_{x_2})^k \psi\right\|^2 \leq C_k \sum_{|\alpha|\leq k} \left\|(1+|\mathbf{x}|^2)^{\frac{k}{2}} \partial_{\mathbf{x}}^\alpha \psi\right\|^2,$$

where for the multi-index $\alpha = (\alpha_1,\alpha_2)$ we used the notation $\partial_{\mathbf{x}}^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$. Hence, in Cartesian coordinates, condition (12) holds if ψ belongs to the weighted Sobolev space $H^{\kappa/2}_{\kappa/2}(\mathbb{R}^2,\mathbb{C}^2) := \{g \in L^2(\mathbb{R}^2,\mathbb{C}^2) \mid (1+|\mathbf{x}|^2)^{\frac{\kappa}{4}} g(\mathbf{x}) \in H^{\kappa/2}(\mathbb{R}^2,\mathbb{C}^2)\}$.

2. Proof of the main result

The strategy of the proof is as follows. We first rewrite the moment in (13) using the decomposition of H as a direct sum of h_j's and the representation $\psi \simeq \oplus\varphi_j$ (see (14)). The main idea consists in taking advantage of the discrete spectrum of the operator h_j and the exponential decay of its eigenfunctions to compensate the growth of the moment. However, the decay is not uniform in j, it turns out (see (29) in Lemma 4.1) that the exponential decay estimate can only be derived outside an interval which grows with j. This is due to the fact that the term $A - m_j/r$ in h_j can be controlled by a fraction of A only outside a ball of radius r_j with $r_j|A(r_j)| \sim |m_j|$. Hence, we split the space $\mathbb{R}^+ \times \mathbb{Z}$ into two regions through the function f, given in Definition 2.2. In the first region, where the values of $r|A(r)|$ are sufficiently large compared to those of j, we control the growth of the moment in r with the exponential decay of the eigenfunctions obtained in Section 4 by an Agmon-type argument. We also need to control the number of these eigenfunctions in the spectral region we consider. This latter bound is derived in Section 3 by using arguments due to Bargmann. In the second region where the values of j are large compared to $r|A(r)|$, the regularity assumption (12) in the angular variable for our initial state yields a decay in the variable j which is used to control the growth of $r^\kappa \lesssim j^\kappa$.

We now turn to the detailed proof of Theorem 1.1.

Since $\mathcal{U}\psi = \oplus_j\varphi_j$ we have

$$\left\||\mathbf{x}|^{\kappa/2}\ \mathrm{e}^{-itH} P_I(H)\psi\right\|^2 = \sum_{j\in\mathbb{Z}} \left\|r^{\kappa/2}\mathrm{e}^{-ith_j} P_I(h_j)\varphi_j\right\|^2. \tag{14}$$

We note that it is enough to consider only the sum for $|j| > J_0$, for sufficiently large $J_0 > 1$. Indeed, let N_j be the number of eigenvalues of h_j in the interval I. Let $\xi_j^{(k)}$, $k = 1, 2, \dots, N_j$ denote the corresponding eigenfunctions of h_j. Then, expanding $P_I(h_j)\varphi_j$ in terms of the $\xi_j^{(k)}$, we have

$$\sum_{|j|\le J_0} \left\| r^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 \le \sum_{|j|\le J_0} \sum_{k=1}^{N_j} \left\| r^{\kappa/2} \xi_j^{(k)} \right\|^2 < \infty.$$

where the last inequality holds in view of Remark 4.2 below. To estimate the right hand side of (14) we split the integration $\sum_j \int_0^\infty$ into two regions characterized by

$$r_j \equiv r_j(\delta_0) := \sup\{r \in \mathbb{R}^+ \mid |m_j| \ge \delta_0 r |A(r)|\}, \tag{15}$$

for some $\delta_0 \in (0,1)$ which is chosen in the proof of Lemma 4.1 below.

Remark 2.1. *Note that since A is continuous and $A(r) \to \infty$ as $r \to \infty$ we have that $r_j < \infty$ and that $r_j \to \infty$ as $|j| \to \infty$. Moreover, the supremum in (15) is attained and hence*

$$|m_j| = \delta_0 r_j |A(r_j)|. \tag{16}$$

Moreover, we note that

$$|m_j| \le \delta_0 r |A(r)|, \quad r \ge r_j. \tag{17}$$

Definition 2.2. *Let $\theta \in C^\infty(\mathbb{R}^+, [0,1])$ with $\theta(r) = 0$ for $r < 1$ and $\theta(r) = 1$ for $r > 2$. We set $f(r,j) = f_j(r) := \theta(r/3r_j)$ and $f_j^c := 1 - f_j$.*

Proof of Theorem 1.1. We have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \left\| r^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 \\ &\le 2 \sum_{j\in\mathbb{Z}} \left(\left\| f_j r^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 + \left\| f_j^c r^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 \right). \end{aligned} \tag{18}$$

We first estimate the second term of the right hand side of (18) using the regularity in the angular variable for the initial state as given by (12). In what follows we pick $J_0 > 1$ so large that $|A(r)| > 1$ for all $r > r_j$ and $|j| > J_0$. In particular, we have that

$$r_j < |m_j|/\delta_0. \tag{19}$$

Using this and the support properties of f_j^c, we have

$$\begin{aligned} &\sum_{|j|>J_0} \left\| f_j^c r^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 \\ &\le \sum_{|j|>J_0} \left\| f_j^c (6r_j)^{\kappa/2} \mathrm{e}^{-ith_j} P_I(h_j)\varphi_j \right\|^2 \\ &\le \sum_{|j|>J_0} (6|m_j|/\delta_0)^\kappa \left\| \mathrm{e}^{-ith_j} P_I(h_j) \right\|^2 \, \|\varphi_j\|^2 < \infty\,, \end{aligned} \tag{20}$$

where we used (12) in the last bound.

We now estimate the first term in the right hand side of (18). For $|j| > J_0$ we compute

$$\left\| f_j r^{\kappa/2} P_I(h_j) \mathrm{e}^{ith_j} \varphi_j \right\|^2 \leq \sup_{\|\phi\|=1} \left(\sum_{k=1}^{N_j} \left\| f_j r^{\kappa/2} \langle \xi_j^{(k)}, \phi \rangle \xi_j^{(k)} \right\| \right)^2$$
$$\leq \sum_{k=1}^{N_j} \left\| f_j r^{\kappa/2} \xi_j^{(k)} \right\|^2 \sup_{\|\phi\|=1} \sum_{l=1}^{N_j} |\langle \xi_j^{(l)}, \phi \rangle|^2$$
$$= \sum_{k=1}^{N_j} \left\| f_j r^{\kappa/2} \xi_j^{(k)} \right\|^2 .$$

Consider the function $\tilde{f}_j$ defined at the beginning of Section 4 below. Since $f_j = f_j \tilde{f}_j$ we have, choosing also $J_0 > J_2$ (see Lemma 4.1)

$$\left\| f_j r^{\kappa/2} \xi_j^{(k)} \right\| \leq \left\| f_j r^{\kappa/2} \mathrm{e}^{-\gamma\varrho} \right\| \left\| \mathrm{e}^{\gamma\varrho} \tilde{f}_j \xi_j^{(k)} \right\| \leq \left\| f_j r^{\kappa/2} \mathrm{e}^{-\gamma\varrho} \right\| \frac{C}{r_j} \mathrm{e}^{\gamma\varrho(2r_j)},$$

where $\rho(r) = \int_0^r |A(s)| \mathrm{d}s$ is the exponential weight defined in Lemma 4.1. Since $r^{\kappa/2}\mathrm{e}^{-\gamma\varrho}$ decays monotonically at infinity, we may choose $J_0 > 1$ to be so large that the supremum of $f_j r^{\kappa/2}\mathrm{e}^{-\gamma\varrho}$ is bounded above by $(3r_j)^{\kappa/2}\mathrm{e}^{-\gamma\varrho(3r_j)}$. Hence

$$\left\| f_j r^{\kappa/2} \xi_j^{(k)} \right\| \leq \frac{C}{r_j} (3r_j)^{\kappa/2} \mathrm{e}^{-\gamma(\varrho(3r_j) - \varrho(2r_j))}.$$

Note that due to (17)

$$\varrho(3r_j) - \varrho(2r_j) = \int_{2r_j}^{3r_j} r|A(r)| \frac{\mathrm{d}r}{r} \geq \frac{|m_j|}{\delta_0} \ln(\tfrac{3}{2}) > \frac{|m_j|}{3\delta_0},$$

for $|j|$ large enough. Thus, using that $r_j \leq |m_j|$ and Lemma 3.1, we get for $|j| > J_2$

$$\left\| f_j r^{\kappa/2} P_I(h_j) \mathrm{e}^{ith_j} \varphi_j \right\|^2 \leq C^2 \sum_{k=1}^{N_j} \left(\mathrm{e}^{-\gamma \frac{|m_j|}{3\delta_0}} (3|m_j|)^{\kappa/2} \right)^2$$
$$\leq 3^\kappa C^2 C_I |m_j|^{\kappa+1} \ln|m_j| \, \mathrm{e}^{-\gamma \frac{2|m_j|}{3\delta_0}} .$$

Since the last bound is summable for $|j| = |m_j - 1/2| > J_0$ we get the expected result. □

3. Estimate on the number of eigenvalues of h_j

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with purely discrete spectrum. We set for an interval $I \subset \mathbb{R}$

$$N_I(T) := \dim P_I(T)\mathcal{H},$$

i.e. $N_I(T)$ denotes the number of eigenvalues of T in I counted with multiplicity.

Lemma 3.1 (Bound on the number of eigenvalues for h_j)**.** *There is a $J_1 > 1$ such that for any $E > 0$ there is a constant $C_E > 0$ so that*

$$N_{[-E,E]}(h_j) \leq C_E |m_j| \ln|m_j|, \quad \text{for } |j| \geq J_1. \tag{21}$$

Proof. We first note that

$$h_j = -i\sigma_2\partial_r + \sigma_1\left(A(r) - \tfrac{m_j}{r}\right) + V(r)$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$. In addition, we obtain by the spectral theorem

$$N_{[-E,\,E]}(h_j) = N_{[0,\,E^2]}(h_j^2).$$

In the sense of quadratic forms on $C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ we obtain for any $\epsilon \in (0,1)$ the estimate

$$\begin{aligned} h_j^2 \geq & (1-\epsilon)\big[-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r} + \sigma_1 A(r)\big]^2 + \big(1-\tfrac{1}{\epsilon}\big)V^2(r) \\ = & (1-\epsilon)\big[\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + A^2(r) - \tfrac{1}{\epsilon}V^2(r) - \sigma_3 A'(r) - \tfrac{2m_j}{r}A(r)\big]. \end{aligned}$$

Let $\delta := (1-\epsilon)/2$. Due to (9) and (10) we have that $\delta A^2 - \sigma_3 A'$ and $\epsilon A^2 - V^2/\epsilon$ are positive outside a large ball B for $\epsilon \in (0,1)$ sufficiently close to 1. Let $C := \|V^2/\epsilon + \sigma_3 A'\|_{L^\infty(B)}$. Then we find

$$h_j^2 \geq (1-\epsilon)\Big[\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + \delta A^2(r) - C - \tfrac{2|m_j|}{r}|A(r)|\Big].$$

We write

$$h_j^2 - E^2 \geq (1-\epsilon)\Big[\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + W_j\Big],$$

where

$$W_j(r) = \delta A^2(r) - C - (1-\epsilon)^{-1}E^2 - \tfrac{2|m_j|}{r}|A(r)|.$$

Let $R_j := r_j(\delta/4)$, where $r_j(\cdot)$ is defined in (15). This yields, $|m_j| \leq \frac{\delta}{4} r|A(r)|$ for all $r > R_j$. Moreover, we may pick J_1 so large that (recall that $R_j \to \infty$ as $|j| \to \infty$)

$$\tfrac{\delta}{2}A^2(r) - C - (1-\epsilon)^{-1}E^2 > 0, \quad \text{for all} \quad r > R_j,\ |j| > J_1.$$

Thus , $W_j \mathbf{1}_{(R_j,\infty)} \geq 0$ and

$$h_j^2 - E^2 \geq (1-\epsilon)\Big[\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + W_j\mathbf{1}_{(0,R_j]}\Big]. \tag{22}$$

Define

$$D_j := \{r \in (0,R_j) \mid |m_j| \geq \tfrac{\delta}{4} r|A(r)|\}. \tag{23}$$

Note that if $r \in (0,R_j) \cap (\mathbb{R}^+ \setminus D_j)$ then $\frac{\delta}{2}A(r)^2 - \frac{2|m_j|}{r}|A(r)| \geq 0$. Hence we have

$$h_j^2 - E^2 \geq (1-\epsilon)\Big[\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + W_j^<\Big], \tag{24}$$

where

$$W_j^<(r) := (\delta A^2(r) - \tfrac{2|m_j|}{r}|A(r)|)\mathbf{1}_{D_j} - (C + (1-\epsilon)^{-1}E^2)\mathbf{1}_{(0,R_j]}. \tag{25}$$

An application of the min-max principle leads to

$$\begin{aligned} N_{[0,E^2]}\big(h_j^2\big) &= N_{(-\infty,0]}\big(h_j^2 - E^2\big) \\ &\leq N_{(-\infty,0]}\big(\big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 + W_j^<\big). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} \big(-i\sigma_2\partial_r - \sigma_1\tfrac{m_j}{r}\big)^2 &= -\partial_r^2 + \tfrac{1}{r^2}m_j(m_j - \sigma_3) \\ &= \begin{pmatrix} -\partial_r^2 + \frac{1}{r^2}\,m_j(m_j-1) & 0 \\ 0 & -\partial_r^2 + \frac{1}{r^2}\,m_j(m_j+1) \end{pmatrix}. \end{aligned}$$

Note that $m_j(m_j \pm 1) > 0$ for $|j| > J_1$. Using the generalized Bargmann estimate [2] (see also [15, Theorem XIII.9]) we obtain for $|m_j| > 1/2$

$$N_{(-\infty,0]}\big(-\partial_r^2 + \tfrac{1}{r^2}\, m_j(m_j - 1) + W_j^<\big) \le \begin{cases} \frac{1}{2m_j-1} \int_0^\infty r|W_j^<(r)|\mathrm{d}r & \text{if } m_j > \frac{1}{2} \\ \frac{1}{2|m_j|+1} \int_0^\infty r|W_j^<(r)|\mathrm{d}r & \text{if } m_j < -\frac{1}{2} \end{cases}$$

and

$$N_{(-\infty,0]}\big(-\partial_r^2 + \tfrac{1}{r^2}\, m_j(m_j + 1) + W_j^<\big) \le \begin{cases} \frac{1}{2m_j+1} \int_0^\infty r|W_j^<(r)|\mathrm{d}r & \text{if } m_j > \frac{1}{2} \\ \frac{1}{2|m_j|-1} \int_0^\infty r|W_j^<(r)|\mathrm{d}r & \text{if } m_j < -\frac{1}{2} \end{cases},$$

and therefore

$$N_{(-\infty,0]}\big(\big(-i\sigma_2\partial_r - \sigma_1 \tfrac{m_j}{r}\big)^2 + W_j^<\big) \le \frac{1}{|m_j| - 1/2} \int_0^\infty r|W_j^<(r)|\mathrm{d}r. \tag{26}$$

Now we estimate using the definition of D_j

$$\begin{aligned} \int_0^\infty r|W_j^<(r)|\mathrm{d}r &\le \frac{(C + (1-\epsilon)^{-1}E^2)R_j^2}{2} + \int_{D_j} r\big|\delta A^2(r) - \tfrac{2|m_j|}{r}|A(r)|\big|\mathrm{d}r \\ &\le \frac{(C + (1-\epsilon)^{-1}E^2)R_j^2}{2} + \int_{D_j} 2|m_j||A(r)|\mathrm{d}r. \end{aligned} \tag{27}$$

Furthermore,

$$\begin{aligned} \int_{D_j} 2|m_j||A(r)|\mathrm{d}r &= \int_{D_j\cap(0,1)} 2|m_j||A(r)|\mathrm{d}r + \int_{D_j\cap(1,\infty)} 2|m_j||A(r)|\mathrm{d}r \\ &\le 2|m_j|\, \|A\|_{L^\infty[0,1]} + \tfrac{8m_j^2}{\delta}\ln(R_j). \end{aligned} \tag{28}$$

Note that in view of Remark 2.1, and the fact that $|A(r)|$ grows at infinity, we have for sufficiently large J_1

$$R_j = \frac{4|m_j|}{\delta|A(R_j)|} \le \frac{4|m_j|}{\delta}, \quad |j| > J_1.$$

This together with (26), (27), and (28) yields the result. $\square$

4. Exponential decay of eigenfunctions of h_j

Let $r_j \equiv r_j(\delta_0)$ be given as in (15). We note that $\delta_0 \in (0,1)$ will be fixed throughout the proof of the next lemma. For the function θ as defined in Definition 2.2, we set $\tilde{f}(r,j) = \tilde{f}_j(r) := \theta(r/r_j)$.

Lemma 4.1. *There exist $\gamma > 0$ and $J_2 > 1$ such that for all $|j| > J_2$ the following holds: Let $\xi_j \in L^2(\mathbb{R}^+, \mathbb{C}^2)$ be a normalized eigenfunction of h_j with energy $E \in I$. Then, for some $C > 0$ (independent of j),*

$$\Big\|A\mathrm{e}^{\gamma\varrho}\tilde{f}_j\xi_j\Big\| \le \frac{C}{r_j}\mathrm{e}^{\gamma\varrho(2r_j)} \tag{29}$$

where for $r \ge 0$, $\varrho(r) := \int_0^r |A(s)|\mathrm{d}s$.

Remark 4.2. *It is clear from the proof that the exponential decay of the eigenfunctions of h_j remains true for $|j| \leq J_2$, however, in this case we get a different constant in front of the exponential.*

Remark 4.3. *Throughout the proof of Lemma 4.1 we use that h_j and $k_j := h_j - V$ are essentially self-adjoint operators on $C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ (see for instance [11] and references therein). Moreover, we also use that V is a perturbation with respect to the magnetic Dirac operator k_j in the sense that $\mathcal{D}(V) \supset C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ and there exists C such that*

$$\|V\varphi\| \leq C(\|k_j\varphi\| + \|\varphi\|) \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2).$$

Indeed, let $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ and χ_R be a smooth characteristic function of a ball of radius $R > 0$. We set $\chi_R^c = 1 - \chi_R$ and $V = V^< + V^>$ where $V^< := V\chi_R$ and $V^> := V\chi_R^c$. We thus have

$$\|V\varphi\| \leq \left\|V^<\varphi\right\| + \left\|V^>\varphi\right\| \leq C_R \|\varphi\| + \|A\chi_R^c\varphi\|,$$

since $V^<$ is bounded and V is dominated by A at infinity according to assumption (9)*. Moreover, for R large enough, we use* (10) *and the identity $k_j^2 = -\partial_r^2 + A_j^2 - \sigma_3 A_j'$ to write*

$$\tfrac{1}{2}\|A\chi_R^c\varphi\| \leq \|k_j\chi_R^c\varphi\| \leq \|(\nabla\chi_R^c)\varphi\| + \|\chi_R^c k_j\varphi\| \leq \|\nabla\chi_R^c\| \|\varphi\| + \|k_j\varphi\|.$$

Proof of Lemma 4.1. In order to derive the Agmon-type estimates, we follow [10]. We set

$$A_j := (A - \frac{m_j}{r}), \tag{30}$$

and we notice that $|A_j| \geq (1 - \delta_0)|A|$ on the support of $\tilde{f}_j$. Let ξ_j be a normalized eigenfunction of h_j associated to an energy E. We define

$$g_j := \mathrm{e}^{\gamma\varrho_\epsilon}\tilde{f}_j\xi_j, \tag{31}$$

where $\gamma \in (0, 1)$ and $\varrho_\epsilon = \dfrac{\varrho}{1 + \epsilon\varrho}$ such that $\varrho(r) = \displaystyle\int_0^r |A(s)|\mathrm{d}s$. Note that ϱ_ϵ is bounded and differentiable. Consider the operator

$$k_j = -i\sigma_2\partial_r + \sigma_1 A_j = h_j - V, \tag{32}$$

and we define

$$Q_j := \mathrm{Re}\langle k_j\mathrm{e}^{\gamma\varrho_\epsilon}g_j, k_j\mathrm{e}^{-\gamma\varrho_\epsilon}g_j\rangle. \tag{33}$$

The task is to obtain bounds for Q_j that will allow us to bound g uniformly in ϵ.

Lower bound. Notice that

$$[k_j, \mathrm{e}^{\gamma\varrho_\epsilon}] = -i\sigma_2\gamma\varrho_\epsilon'\mathrm{e}^{\gamma\varrho_\epsilon}, \tag{34}$$

so that we rewrite

$$\begin{aligned} Q_j &= \mathrm{Re}\langle \mathrm{e}^{-\gamma\varrho_\epsilon}k_j\mathrm{e}^{\gamma\varrho_\epsilon}g_j, \mathrm{e}^{\gamma\varrho_\epsilon}k_j\mathrm{e}^{-\gamma\varrho_\epsilon}g_j\rangle \\ &= \mathrm{Re}\langle (k_j - i\gamma\varrho_\epsilon'\sigma_2)g_j, (k_j + i\gamma\varrho_\epsilon'\sigma_2)g_j\rangle = \|k_jg_j\|^2 - \gamma^2\|\varrho_\epsilon'g_j\|^2. \end{aligned}$$

Moreover, we have

$$k_j^2 = -\partial_r^2 + A_j^2 - \sigma_3 A_j'. \tag{35}$$

In view of Remark 2.1 and (10) for any $\tilde{\epsilon} > 0$ there exists $J_{\tilde{\epsilon}} > 0$ such that for all $|j| > J_{\tilde{\epsilon}}$ one has

$$\langle g_j, A_j'g_j\rangle \leq \tilde{\epsilon}\langle g_j, A^2g_j\rangle$$

and therefore

$$\langle g_j, (A_j^2 - \sigma_3 A_j')g_j\rangle \geq ((1-\delta_0)^2 - \tilde{\epsilon})\langle g_j, A^2 g_j\rangle. \tag{36}$$

Now we drop the term $-\partial_r^2$ of (35). This together with (36) yields

$$\begin{aligned} Q_j &\geq ((1-\delta_0)^2 - \tilde{\epsilon}) \|Ag_j\|^2 - \gamma^2 \|\varrho_\epsilon' g_j\|^2 \\ &\geq ((1-\delta_0)^2 - \tilde{\epsilon}) \|Ag_j\|^2 - \gamma^2 \|\varrho' g_j\|^2 = ((1-\delta_0)^2 - \tilde{\epsilon} - \gamma^2) \|Ag_j\|^2 . \end{aligned} \tag{37}$$

Upper bound. We rewrite

$$\begin{aligned} Q_j &= \mathrm{Re}\langle k_j \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \tilde{f}_j(E-V)\xi_j\rangle + \mathrm{Re}\langle k_j \mathrm{e}^{\gamma\varrho_\epsilon} g_j, -i\sigma_2 \tilde{f}_j'\xi_j\rangle \\ &= \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \tilde{f}_j(E-V)^2\xi_j\rangle + \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \left[k_j, \tilde{f}_j(E-V)\right]\xi_j\rangle \\ &\quad + \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, -i\sigma_2 \tilde{f}_j'(E-V)\xi_j\rangle + \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \left[k_j, -i\sigma_2 \tilde{f}_j'\right]\xi_j\rangle \\ &= \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \tilde{f}_j(E-V)^2\xi_j\rangle + \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \left[k_j, -i\sigma_2 \tilde{f}_j'\right]\xi_j\rangle, \end{aligned}$$

since

$$\mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, -i\sigma_2 \tilde{f}_j'(E-V)\xi_j\rangle = \mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, [V, k_j]\, \tilde{f}_j\xi_j\rangle = 0.$$

Furthermore, we use

$$\mathrm{Re}\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \tilde{f}_j(E-V)^2\xi_j\rangle = \|(E-V)\ g_j\|^2 .$$

In addition, we find some $C > 0$ such that

$$\begin{aligned} |\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, \left[k_j, -i\sigma_2 \tilde{f}_j'\right]\xi_j\rangle| &= |\langle \mathrm{e}^{\gamma\varrho_\epsilon} g_j, (-\tilde{f}_j'' + 2\sigma_3 A_j \tilde{f}_j')\xi_j\rangle| \\ &\leq C\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(\tfrac{1}{r_j}\|g_j\| + \|A_j g_j\|) \\ &\leq C\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(\|g_j\| + (1+\delta_0)\|Ag_j\|), \end{aligned}$$

where in the last inequality we use that $r_j > 1$ (for sufficiently large $|j|$) and (30) together with the support properties of $\tilde{f}_j$. We thus get

$$Q_j \leq \|(E-V)\ g_j\|^2 + C\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(\|g_j\| + (1+\delta_0)\|Ag_j\|). \tag{38}$$

Then, combining (37) and (38) we get for $|j| > J_{\tilde{\epsilon}}$

$$\langle g_j, (((1-\delta_0)^2 - \tilde{\epsilon} - \gamma^2)A^2 - (E-V)^2)g_j\rangle \leq C\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(\|g_j\| + (1+\delta_0)\|Ag_j\|). \tag{39}$$

According to (9) and Remark 2.1 we may pick $\delta_0, \tilde{\epsilon}$ and γ so small that there are constants $J_{\delta_0,\tilde{\epsilon},\gamma}, c_{\delta_0,\tilde{\epsilon},\gamma} > 0$ such that $|A| > 1$ on the support of $\tilde{f}_j$ and, for all $|j| > J_{\delta_0,\tilde{\epsilon},\gamma}$,

$$\langle g_j, [((1-\delta_0)^2 - \tilde{\epsilon} - \gamma^2)A^2 - (E-V)^2]g_j\rangle \geq c_{\delta_0,\tilde{\epsilon},\gamma}\|Ag_j\|^2 . \tag{40}$$

This together with (39) yields

$$\|Ag_j\| \leq \frac{C}{c_{\delta_0,\tilde{\epsilon},\gamma}}\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(\|g_j\|/\|Ag_j\| + (1+\delta_0)) \leq \frac{C}{c_{\delta_0,\tilde{\epsilon},\gamma}}\frac{\mathrm{e}^{\gamma\varrho(2r_j)}}{r_j}(2+\delta_0). \tag{41}$$

The claim follows using the theorem of monotonic convergence for the limit $\epsilon \to 0$ of (41). $\square$

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